377

PROVE that the rectangle under the focal distances of the origin in the conic represented by the equation $ax^2 + 2hxy + by^2 = 2y$, is $R = 1/(ab - h^2)$.

[*F. P. Matz.*]

SOLUTION.

The foci are given by

$$(Cx + h)^2 = \tfrac{1}{2}\varDelta(R + a - b), \quad (Cy - a)^2 = \tfrac{1}{2}\varDelta(R + b - a),$$

(See Salmon's Conic Sections, p. 239, exercise and Art. 151) where $C = ab - h^2$, $R^2 = 4h^2 + (a - b)^2$, and $\varDelta$, for this exercise, $= -a$.

$$\therefore (Cx + h)^2 = \tfrac{1}{2}a(b - a - R) = A^2,$$

$$(Cy - a)^2 = \tfrac{1}{2}a(a - b - R) = B^2;$$

$$\therefore x = \frac{-h \pm A}{C}, \quad y = \frac{a \mp B}{C};$$

$$\therefore R = \frac{1}{C^2}\sqrt{\{(A + h)^2 + (a + B)^2\}\{(A - h)^2 + (a - B)^2\}}$$

$$= \frac{1}{C} = \frac{1}{ab - h^2}.$$

[*G. B. M. Zerr.*]

Also solved by William Hoover and W. O. Whitescarver.

EXERCISES.

379

FIND the radius of a circle circumscribing the three tangent-circles of radii a, b, and c, respectively. [*F. P. Matz.*]

380

THE sides of a variable rectangle pass through four fixed points. Find the position of the rectangle and its dimensions when its area is a maximum.

[*Geo. R. Dean.*]

381

FROM a point in the circumference of a circle of radius R as centre is described the external arc of a circle of radius r. Determine r so that the area of the lune shall equal that of the original circle. [*W. M. Thornton.*]

382

FOUR equal circles tangent to each other cut off equal areas from a given circle. Required the radii of the cutting circles when the aggregate area cut off from the given circle is the greatest possible.   [*Artemas Martin.*]

383

IF c', c'', c''' be the sides of any triangle inscribed in an ellipse, and b', b'', b''' the semi-diameters parallel to the sides, show that the area is

$$A = abc'c''c'''/(4b'b''b''') . \quad [\ W.\ O.\ Whitescarver.]$$

384

IF c be a chord of an ellipse through the points whose eccentric angles are α and β, and b' the semi-diameter parallel to the chord, show that the area of the triangle formed by the chord and the tangents at its extremities is

$$S = \frac{abc^2}{4b'^2} \tan \tfrac{1}{2} (\alpha - \beta) . \quad [\ W.\ O.\ Whitescarver.]$$

385

IF in exercise 384 a tangent be drawn parallel to the chord, show that the base of the triangle formed will be $c \sec \tfrac{1}{2} (\alpha - \beta)$, and its area $ab \tan^3 \tfrac{1}{2} (\alpha - \beta)$.   [*W. O. Whitescarver.*]

386

PROVE that the curvature of a bicircular quartic at a point x is the arithmetic mean of the curvature of the four circles which touch the curve at x and pass through the respective real foci.   [*F. Morley.*]

387

IT is desirable to have a linkwork or other mechanism for keeping four points in the shape of a variable harmonic tetrad, i. e. at the ends of harmonic chords of an arbitrary circle.   [*F. Morley.*]

388

AN observer, whose eye is at a distance e above the surface of a pond, notices that the setting sun touches a hill-top, distant t from the observer, just as the last of its watery image disappears behind the hill-top reflected in the pond. How high is the hill above the surface of the pond? Give solutions adapted to two cases, (1) when heights are small in comparison with horizontal distances, and (2) when they are not.   [*R. A. Harris.*]

389

A PORTION of xyz space is transformed by means of a rational integral function, algebraic or transcendental; show that ordinarily the XY, XZ, YZ projections of transformed angles which lay originally in planes parallel to

xy, xz, yz, respectively, measure the same as did the original angles—provided the measurement of angles in, or parallel to, YZ or yz be made with an hyperbolic protractor.   [*R. A. Harris.*]

390

A VIBRATION is made up of two simple harmonic components, taken in the same direction, whose periods are as $1:2$, and whose amplitudes are such that the number of maxima and minima of the resultant is the same as of the component of shorter period ; show that the mean amplitude of the resultant is nearly independent of what phase of the one component falls upon a given phase of the other.   [*R. A. Harris.*]

391

Two simple harmonic components, taken in the same direction, are combined into one vibration. Show how to determine, by graphic means, the position in time (or angle) of any resultant maximum with reference to the neighboring maximum of the component of shorter period—the relative lengths of the periods being, for successive cases, $1:1, 1:2$, and, if possible, $1:3$ or more.   [*R. A. Harris.*]

392

A SYSTEM of great circles intersects upon the equator of a sphere ; a curve is drawn connecting points on the spherical surface where the circles of this system make a constant angle a with the meridians. Show that the stereographic projection of any such curve is a circular cubic whose equation may be written

$$\tan a \tan \theta = \frac{c^2 + r^2}{c^2 - r^2},$$

c being the radius of the sphere, and one of the poles being the centre of the projection.   [*R. A. Harris.*]

393

SHOW that if y be a quadratic function of x between the limits 0, h, its mean value can be expressed in an infinite number of ways by the formula, $\lambda y_1 + (1 - \lambda) y_2$, where y_1, y_2 correspond to the values

$$x_1 = \frac{h}{2} - h \sqrt{\frac{1}{12} \frac{1 - \lambda}{\lambda}}, \quad x_2 = \frac{h}{2} + h \sqrt{\frac{1}{12} \frac{\lambda}{1 - \lambda}}.$$

[*Wm. M. Thornton.*]

394

A HETEROGENEOUS rod is hung from a fixed point by two elastic threads of given length fastened at its extremities. Find the position of equilibrium.

[*W. H. Echols.*]

395

FIND the difference between the specific heats of a gas whose equation is

$$pv = R\tau - a/\tau v.$$ [*F. P. Matz.*]